Conservation laws for invariant functionals containing compositions*

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Abstract

The study of problems of the calculus of variations with compositions is a quite recent subject with origin in dynamical systems governed by chaotic maps. Available results are reduced to a generalized Euler-Lagrange equation that contains a new term involving inverse images of the minimizing trajectories. In this work we prove a generalization of the necessary optimality condition of DuBois-Reymond for variational problems with compositions. With the help of the new obtained condition, a Noether-type theorem is proved. An application of our main result is given to a problem appearing in the chaotic setting when one consider maps that are ergodic.

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1 Introduction and motivation

The theory of variational calculus for problems with compositions has been recently initiated in [5]. The new theory considers integral functionals that depend not only on functions $q(\cdot)$ and their derivatives $\dot{q}(\cdot)$, but also on compositions $(q \circ q)(\cdot)$ of $q(\cdot)$ with $q(\cdot)$. As far as chaos is often a byproduct of iteration

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of nonlinear maps [2], such problems serve as an interesting model for chaotic dynamical systems. Let us briefly review this relation (for more details, we refer the interested reader to [3, 4, 5]). Let $q : [0,1] \to [0,1]$ be a piecewise monotonic map with probability density function $f_q(\cdot)$, which captures the long term statistical behavior of a nonlinear dynamical system. It is natural (see [2]) to consider the problem of minimizing or maximizing the functional

$$I[q(\cdot), f_q(\cdot)] = \int_0^1 \left(q(t) - t\right)^2 f_q(t) dt \,, \tag{1}$$

which depends on $q(\cdot)$ and its probability density function $f_q(\cdot)$ (usually a complicated function of $q(\cdot)$). It turns out that $f_q(\cdot)$ is the fixed point of the Frobenius-Perron operator $P_q[\cdot]$ associated with $q(\cdot)$. For a piecewise monotonic map $q : [0,1] \to [0,1]$ with r pieces, $P_q[\cdot]$ has the representation

$$P_q[f](t) = \sum_{v \in \{q^{-1}(t)\}} \frac{f(v)}{|\dot{q}(v)|} \,,$$

where for any point $t \in [0,1]$ the set $\{q^{-1}(t)\}$ consists of at most r points. The fixed point $f_q(\cdot)$ associated with an ergodic map $q(\cdot)$ can be expressed as the limit

$$f_q = \lim_{n \to \infty} \sum_{i=0}^{n-1} P_q^i[\mathbf{1}] \,, \tag{2}$$

where $\mathbf{1}$ is the constant function 1 on $[0,1]$. Substituting (2) into (1), and using the adjoint property [2, Prop. 4.2.6], one eliminates the probability density function $f_q(\cdot)$, obtaining (1) in the form

$$I[q(\cdot)] = \int_0^1 L\left(t, q(t), q^{(2)}(t), q^{(3)}(t), \ldots\right) dt \,,$$

where we are using the notation $q^{(i)}(\cdot)$ to denote the i-th composition of $q(\cdot)$ with itself: $q^{(1)}(t) = q(t)$, $q^{(2)}(t) = (q \circ q)(t)$, $q^{(3)}(t) = (q \circ q \circ q)(t)$, etc. In [5] a generalized Euler-Lagrange equation, which involves the inverse images of the extremizing function $q(\cdot)$ (cf. (11)), was proved for such functionals in the cases

$$\int_a^b L\left(t, q(t), q^{(2)}(t)\right) dt \,,$$

$$\int_a^b L\left(t, q(t), \dot{q}(t), q^{(2)}(t)\right) dt \,,$$

or

$$\int_a^b L\left(t, q(t), q^{(2)}(t), q^{(3)}(t)\right) dt \,.$$

To the best of our knowledge, these generalized Euler-Lagrange equations comprise all the available results on the subject. Thus, one concludes that the

theory of variational calculus with compositions is in its childhood: much remains to be done. Here we go a step further in the theory of functionals containing compositions. We are mainly interested in Noether's classical theorem, which is one of the most beautiful results of the calculus of variations and optimal control, with many important applications in Physics (see e.g. [6, 13, 14]), Economics (see e.g. [1, 17]), and Control Engineering (see e.g. [11, 15, 18, 20, 22]), and source of many recent extensions and developments (see e.g. [7, 8, 9, 10, 16, 19, 21]). Noether's symmetry theorem describes the universal fact that invariance with respect to some family of parameter transformations gives rise to the existence of certain conservation laws, i.e. expressions preserved along the Euler-Lagrange or Pontryagin extremals of the problem. Our results are a generalized DuBois-Reymond necessary optimality condition (Theorem 7), and a generalized Noether's theorem (Theorem 13) for functionals of the form $\int_a^b L\left(t, q(t), \dot{q}(t), q^{(2)}(t)\right) dt$. In §4 an illustrative example is presented.

2 Preliminaries – review of classical results of the calculus of variations

There exist many different ways to prove the classical Noether's theorem (cf. e.g. [6, 12, 13, 17]). We review here one of those proofs, which is based on the DuBois-Reymond necessary condition. Although this proof is not so common in the literature of Noether's theorem, it turns out to be the most suitable approach when dealing with functionals containing compositions.

Let us consider the fundamental problem of the calculus of variations:

$$I[q(\cdot)] = \int_a^b L\left(t, q(t), \dot{q}(t)\right) dt \longrightarrow \min \tag{P}$$

under the boundary conditions $q(a) = q_a$ and $q(b) = q_b$, where $\dot{q} = \frac{dq}{dt}$, with $q(\cdot)$ a piecewise-smooth function, and the Lagrangian $L : [a, b] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a C^2 function with respect to all its arguments.

The concept of symmetry has a very important role in mathematics and its applications. Symmetries are defined through transformations of the system that leave the problem *invariant*.

Definition 1 (Invariance of (P))**.** *The integral functional* (P) *it said to be invariant under the ε-parameter infinitesimal transformations*

$$\begin{cases} \bar{t} = t + \varepsilon\tau(t, q) + o(\varepsilon)\,, \\ \bar{q}(t) = q(t) + \varepsilon\xi(t, q) + o(\varepsilon)\,, \end{cases} \tag{3}$$

where τ and ξ are piecewise-smooth, if

$$\int_{t_a}^{t_b} L\left(t, q(t), \dot{q}(t)\right) dt = \int_{\bar{t}(t_a)}^{\bar{t}(t_b)} L\left(\bar{t}, \bar{q}(\bar{t}), \dot{\bar{q}}(\bar{t})\right) d\bar{t} \tag{4}$$

for any subinterval $[t_a, t_b] \subseteq [a, b]$*.*

Along the work we denote by $\partial_i L$ the partial derivative of L with respect to its i-th argument.

Theorem 2 (Necessary condition of invariance)**.** *If functional* (P) *is invariant under the infinitesimal transformations* (3)*, then*

$$\partial_1 L\left(t, q, \dot{q}\right) \tau + \partial_2 L\left(t, q, \dot{q}\right) \cdot \xi + \partial_3 L\left(t, q, \dot{q}\right) \cdot \left(\dot{\xi} - \dot{q}\dot{\tau}\right) + L\left(t, q, \dot{q}\right) \dot{\tau} = 0\,. \quad (5)$$

Proof. Since (4) is to be satisfied for any subinterval $[t_a, t_b] \subseteq [a, b]$, equality (4) is equivalent to

$$\left[L\left(t + \varepsilon\tau + o(\varepsilon), q + \varepsilon\xi + o(\varepsilon), \frac{\dot{q} + \varepsilon\dot{\xi} + o(\varepsilon)}{1 + \varepsilon\dot{\tau} + o(\varepsilon)}\right)\right] \frac{d\bar{t}}{dt} = L\left(t, q, \dot{q}\right)\,. \quad (6)$$

We obtain (5) differentiating both sides of (6) with respect to ε, and then setting $\varepsilon = 0$. □

Another very important notion in mathematics and its applications is the concept of *conservation law*. One of the most important conservation laws was proved by Leonhard Euler in 1744: when the Lagrangian $L(q, \dot{q})$ corresponds to a system of conservative points, then

$$-\,L\left(q(t), \dot{q}(t)\right) + \frac{\partial L}{\partial \dot{q}}\left(q(t), \dot{q}(t)\right) \cdot \dot{q}(t) \equiv \text{constant}\,, \quad (7)$$

$t \in [a, b]$, holds along the solutions of the Euler-Lagrange equations.

Definition 3 (Conservation law)**.** *A quantity* $C(t, q, \dot{q})$ *defines a conservation law if*

$$\frac{d}{dt} C(t, q(t), \dot{q}(t)) = 0\,, \quad t \in [a, b]\,,$$

along all the solutions $q(\cdot)$ *of the Euler-Lagrange equation*

$$\frac{d}{dt}\partial_3 L\left(t, q, \dot{q}\right) = \partial_2 L\left(t, q, \dot{q}\right)\,. \quad (8)$$

Conservation laws can be used to lower the order of the Euler-Lagrange equations (8) and simplify the resolution of the respective problems of the calculus of variations and optimal control [16]. Emmy Amalie Noether formulated in 1918 a very general principle on conservation laws, with many important implications in modern physics, economics and engineering. Noether's principle asserts that *"the invariance of the functional* $\int_a^b L\left(t, q(t), \dot{q}(t)\right) dt$ *under one-parameter infinitesimal transformations* (3)*, imply the existence of a conservation law".* One particular example of application of Noether's theorem gives (7), which corresponds to conservation of energy in classical mechanics or to the income-wealth law of economics.

Theorem 4 (Noether's theorem)**.** *If functional* (P) *is invariant, in the sense of the Definition 1, then*

$$C(t,q,\dot{q}) = \partial_3 L\left(t,q,\dot{q}\right)\cdot\xi(t,q) + \left(L(t,q,\dot{q}) - \partial_3 L\left(t,q,\dot{q}\right)\cdot\dot{q}\right)\tau(t,q) \tag{9}$$

defines a conservation law.

We recall here the proof of Theorem 4 by means of the classical necessary optimality condition of DuBois-Reymond.

Theorem 5 (DuBois-Reymond condition)**.** *If* $q(\cdot)$ *is a solution of problem* (P)*, then*

$$\partial_1 L\left(t,q,\dot{q}\right) = \frac{d}{dt}\left[L\left(t,q,\dot{q}\right) - \partial_3 L\left(t,q,\dot{q}\right)\cdot\dot{q}\right]\,. \tag{10}$$

Proof. The DuBois-Reymond necessary optimality condition is easily proved using the Euler-Lagrange equation (8):

$$\begin{aligned}
&\frac{d}{dt}\left[L\left(t,q,\dot{q}\right) - \partial_3 L\left(t,q,\dot{q}\right)\cdot\dot{q}\right]\\
&\quad= \partial_1 L\left(t,q,\dot{q}\right) + \partial_2 L\left(t,q,\dot{q}\right)\cdot\dot{q} + \partial_3 L\left(t,q,\dot{q}\right)\cdot\ddot{q}\\
&\qquad - \frac{d}{dt}\partial_3 L\left(t,q,\dot{q}\right)\cdot\dot{q} - \partial_3 L\left(t,q,\dot{q}\right)\cdot\ddot{q}\\
&\quad= \partial_1 L\left(t,q,\dot{q}\right) + \dot{q}\cdot\left(\partial_2 L\left(t,q,\dot{q}\right) - \frac{d}{dt}\partial_3 L\left(t,q,\dot{q}\right)\right)\\
&\quad= \partial_1 L\left(t,q,\dot{q}\right)\,.
\end{aligned}$$

□

Proof. (of Theorem 4) To prove the Noether's theorem, we use the Euler-Lagrange equation (8) and the DuBois-Reymond condition (10) into the necessary condition of invariance (5):

$$\begin{aligned}
0 &= \partial_1 L\left(t,q,\dot{q}\right)\tau + \partial_2 L\left(t,q,\dot{q}\right)\cdot\xi\\
&\quad + \partial_3 L\left(t,q,\dot{q}\right)\cdot\left(\dot{\xi} - \dot{q}\dot{\tau}\right) + L\left(t,q,\dot{q}\right)\dot{\tau}\\
&= \partial_2 L\left(t,q,\dot{q}\right)\cdot\xi + \partial_3 L\left(t,q,\dot{q}\right)\cdot\dot{\xi} + \partial_1 L\left(t,q,\dot{q}\right)\tau\\
&\quad + \dot{\tau}\left(L\left(t,q,\dot{q}\right) - \partial_3 L\left(t,q,\dot{q}\right)\cdot\dot{q}\right)\\
&= \frac{d}{dt}\partial_3 L\left(t,q,\dot{q}\right)\cdot\xi + \partial_3 L\left(t,q,\dot{q}\right)\cdot\dot{\xi}\\
&\quad + \frac{d}{dt}\left(L\left(t,q,\dot{q}\right) - \partial_3 L\left(t,q,\dot{q}\right)\cdot\dot{q}\right)\tau\\
&\quad + \dot{\tau}\left(L\left(t,q,\dot{q}\right) - \partial_3 L\left(t,q,\dot{q}\right)\cdot\dot{q}\right)\\
&= \frac{d}{dt}\left[\partial_3 L\left(t,q,\dot{q}\right)\cdot\xi + \left(L(t,q,\dot{q}) - \partial_3 L\left(t,q,\dot{q}\right)\cdot\dot{q}\right)\tau\right]\,.
\end{aligned}$$

□

3 Main results

We consider the following problem of the calculus of variations with composition of functions:

$$I[q(\cdot)] = \int_a^b L\left(t, q(t), \dot{q}(t), z(t)\right) dt \longrightarrow \min \tag{P_c}$$

subject to given boundary conditions $q(a) = q_a$, $q(b) = q_b$, $z(a) = z_a$, and $z(b) = z_b$, where $\dot{q} = \frac{dq}{dt}$ and $z(t) = (q \circ q)(t)$. We assume that the Lagrangian $L : [a, b] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a function of class C^2 with respect to all the arguments, and that admissible functions $q(\cdot)$ are piecewise-smooth. The main result of [5] is an extension of the Euler-Lagrange equation (8) for problems of the calculus of variations (P_c).

Theorem 6 ([5])**.** *If $q(\cdot)$ is a weak minimizer of problem (P_c), then $q(\cdot)$ satisfies the* Euler-Lagrange equation

$$\begin{aligned} &\partial_2 L\left(x, q(x), \dot{q}(x), z(x)\right) - \frac{d}{dx}\partial_3 L\left(x, q(x), \dot{q}(x), z(x)\right) \\ &\quad + \partial_4 L\left(x, q(x), \dot{q}(x), z(x)\right)\dot{q}(q(x)) + \sum_{t=q^{-1}(x)} \frac{\partial_4 L\left(t, q(t), \dot{q}(t), z(t)\right)}{|\dot{q}(t)|} = 0 \end{aligned} \tag{11}$$

for any $x \in (a, b)$.

3.1 Generalized DuBois-Reymond condition

We begin by proving an extension of the DuBois-Reymond necessary optimality condition (10) for problems of the calculus of variations (P_c).

Theorem 7 (cf. Theorem 5)**.** *If $q(\cdot)$ is a weak minimizer of problem (P_c), then $q(\cdot)$ satisfies the* DuBois-Reymond condition

$$\begin{aligned} &\frac{d}{dx}\Big[L\left(x, q(x), \dot{q}(x), z(x)\right) - \partial_3 L\left(x, q(x), \dot{q}(x), z(x)\right)\dot{q}(x)\Big] \\ &\qquad = \partial_1 L\left(x, q(x), \dot{q}(x), z(x)\right) - \dot{q}(x) \sum_{t=q^{-1}(x)} \frac{\partial_4 L\left(t, q(t), \dot{q}(t), z(t)\right)}{|\dot{q}(t)|} \end{aligned} \tag{12}$$

for any $x \in (a, b)$.

Remark 8. *If $L\left(t, q, \dot{q}, z\right) = L\left(t, q, \dot{q}\right)$, then* (12) *coincides with the classical DuBois-Reymond condition* (10)*.*

Proof. To prove Theorem 7 we use the Euler-Lagrange equation (11):

$$\begin{aligned}
&\frac{d}{dx}\Big[L\left(x,q,\dot{q},z\right)-\partial_3 L\left(x,q,\dot{q},z\right)\dot{q}\Big]\\
&\quad=\partial_1 L\left(x,q,\dot{q},z\right)+\partial_2 L\left(x,q,\dot{q},z\right)\dot{q}\\
&\qquad+\partial_3 L\left(x,q,\dot{q},z\right)\ddot{q}+\partial_4 L\left(x,q,\dot{q},z\right)\dot{q}(q(x))\dot{q}\\
&\qquad-\dot{q}\frac{d}{dx}\partial_3 L\left(x,q,\dot{q},z\right)-\partial_3 L\left(x,q,\dot{q},z\right)\ddot{q}\\
&\quad=\partial_1 L\left(x,q,\dot{q},z\right)+\dot{q}\Big(\partial_2 L\left(x,q,\dot{q},z\right)\\
&\qquad+\partial_4 L\left(x,q,\dot{q},z\right)\dot{q}(q(x))-\frac{d}{dx}\partial_3 L\left(x,q,\dot{q},z\right)\Big)\\
&\quad=\partial_1 L\left(x,q,\dot{q},z\right)-\dot{q}(x)\sum_{t=q^{-1}(x)}\frac{\partial_4 L\left(t,q(t),\dot{q}(t),z(t)\right)}{|\dot{q}(t)|}\,.
\end{aligned}$$

□

3.2 Noether's theorem for functionals containing compositions

We introduce now the definition of *invariance* for the functional (P_c). As done in the proof of Theorem 2 (see (6)), we get rid off of the integral signs in (4).

Definition 9 (cf. Definition 1)**.** *We say that functional* (P_c) *is invariant under the infinitesimal transformations* (3) *if*

$$L\left(\bar{t},\bar{q}(\bar{t}),\bar{q}'(\bar{t}),\bar{z}(\bar{t})\right)\frac{d\bar{t}}{dt}=L\left(t,q(t),\dot{q}(t),z(t)\right)+o(\varepsilon)\,, \tag{13}$$

where $\bar{q}'=d\bar{q}/d\bar{t}$*.*

Along the work, in order to simplify the presentation, we sometimes omit the arguments of the functions.

Theorem 10 (cf. Theorem 2)**.** *If functional* (P_c) *is invariant under the infinitesimal transformations* (3)*, then*

$$\begin{aligned}
\partial_1 L\left(t,q,\dot{q},z\right)\tau+\partial_2 L\left(t,q,\dot{q},z\right)\xi+\partial_3 L\left(t,q,\dot{q},z\right)\left(\dot{\xi}-\dot{q}\dot{\tau}\right)&\\
+\partial_4 L\left(t,q,\dot{q},z\right)\dot{q}(q(t))\xi+\partial_4 L\left(t,q,\dot{q},z\right)\xi(q(t))+L\dot{\tau}=0\,.&
\end{aligned} \tag{14}$$

Proof. Equation (13) is equivalent to

$$\begin{aligned}
L\Bigg(&t+\varepsilon\tau+o(\varepsilon),q+\varepsilon\xi+o(\varepsilon),\frac{\dot{q}+\varepsilon\dot{\xi}+o(\varepsilon)}{1+\varepsilon\dot{\tau}+o(\varepsilon)},\\
&q(q+\varepsilon\xi+o(\varepsilon))+\varepsilon\xi(q+\varepsilon\xi+o(\varepsilon))\Bigg)\times\left(1+\varepsilon\dot{\tau}+o(\varepsilon)\right)\\
&=L\left(t,q,\dot{q},z\right)+o(\varepsilon)\,.
\end{aligned} \tag{15}$$

We obtain equation (14) differentiating both sides of equality (15) with respect to the parameter ε, and then setting $\varepsilon = 0$. $\square$

Remark 11. *Using the Frobenius-Perron operator (see [2, Chap. 4]) and the Euler-Lagrange equation* (11)*, we can write* (14) *in the following form:*

$$\begin{aligned}
&\partial_1 L\left(x, q, \dot{q}, z\right) \tau + \partial_2 L\left(x, q, \dot{q}, z\right) \xi \\
&\qquad + \partial_3 L\left(x, q, \dot{q}, z\right) \left(\dot{\xi} - \dot{q}\dot{\tau}\right) + \partial_4 L\left(x, q, \dot{q}, z\right) \dot{q}(q(x))\xi \\
&\qquad\quad + \sum_{t=q^{-1}(x)} \frac{\partial_4 L\left(t, q(t), \dot{q}(t), z(t)\right)}{|\dot{q}(t)|}\xi + L\dot{\tau} \\
&\qquad\quad = \partial_1 L\left(x, q, \dot{q}, z\right) \tau + \frac{d}{dx}\partial_3 L\left(x, q, \dot{q}, z\right) \xi \\
&\qquad\qquad + \partial_3 L\left(x, q, \dot{q}, z\right) \left(\dot{\xi} - \dot{q}\dot{\tau}\right) + L\dot{\tau} = 0\,. \qquad (16)
\end{aligned}$$

Definition 12 (Conservation law for (P_c))**.** *We say that a quantity* $C\left(x, q, \dot{q}, z\right)$ *defines a* conservation law for functionals containing compositions *if*

$$\frac{d}{dx}C\left(x, q(x), \dot{q}(x), z(x)\right) = 0$$

along all the solutions $q(\cdot)$ *of the Euler-Lagrange equation* (11)*.*

Our main result is an extension of Noether's theorem for problems of the calculus of variations (P_c) containing compositions.

Theorem 13 (Noether's theorem for (P_c))**.** *If functional* (P_c) *is invariant, in the sense of the Definition 9, and there exists a function* $f = f(x, q, \dot{q}, z)$ *such that*

$$\frac{df}{dx}\left(x, q(x), \dot{q}(x), z(x)\right) = \tau\dot{q}(x) \sum_{t=q^{-1}(x)} \frac{\partial_4 L\left(t, q(t), \dot{q}(t), z(t)\right)}{|\dot{q}(t)|}\,, \qquad (17)$$

then

$$\begin{aligned}
&C\left(x, q(x), \dot{q}(x), z(x)\right) \\
&\qquad = \Big[L(x, q(x), \dot{q}(x), z(x)) - \partial_3 L\left(x, q(x), \dot{q}(x), z(x)\right) \dot{q}\Big] \tau(x, q) \\
&\qquad\qquad + \partial_3 L\left(x, q(x), \dot{q}(x), z(x)\right) \xi(x, q) + f(x, q(x), \dot{q}(x), z(x)) \qquad (18)
\end{aligned}$$

defines a conservation law (Definition 12).

Remark 14. *If* $L\left(x, q, \dot{q}, z\right) = L\left(x, q, \dot{q}\right)$*, then* f *is a constant and expression* (18) *is equivalent to the conserved quantity* (9) *given by the classical Noether's theorem.*

Proof. To prove the theorem, we use conditions (12) and (17) in (16):

$$\begin{aligned}
0 &= \partial_1 L\left(x,q,\dot{q},z\right)\tau + \frac{d}{dx}\partial_3 L\left(x,q,\dot{q},z\right)\xi \\
&\quad + \partial_3 L\left(x,q,\dot{q},z\right)\left(\dot{\xi} - \dot{q}\dot{\tau}\right) + L\dot{\tau} \\
&= \tau\frac{d}{dx}\Big[L\left(x,q,\dot{q},z\right) - \partial_3 L\left(x,q,\dot{q}),z)\right)\dot{q}\Big] \\
&\quad + \dot{\tau}\left[L\left(x,q,\dot{q},z\right) - \partial_3 L\left(x,q,\dot{q}),z)\right)\dot{q}\right] \\
&\quad + \dot{\xi}\partial_3 L\left(x,q,\dot{q},z\right) + \xi\frac{d}{dx}\partial_3 L\left(x,q,\dot{q},z\right) \\
&\quad + \tau\dot{q}(x)\sum_{t=q^{-1}(x)}\frac{\partial_4 L\left(t,q(t),\dot{q}(t),z(t)\right)}{|\dot{q}(t)|} \\
&= \frac{d}{dx}\Bigg\{\partial_3 L\left(x,q,\dot{q},z\right)\xi + \Big[L\left(x,q,\dot{q},z\right) \\
&\quad - \partial_3 L\left(x,q,\dot{q},z\right)\dot{q}\Big]\tau + f(x,q,\dot{q},z)\Bigg\}.
\end{aligned}$$

□

4 An example

Let us consider the problem

$$\begin{gathered}
I[q(\cdot)] = \frac{1}{3}\int_0^1 \left[x + q(x) + q(q(x))\right]dx \longrightarrow \min \\
q(0) = 1\,, \quad q(1) = 0\,, \\
q(q(0)) = 0\,, \quad q(q(1)) = 1\,.
\end{gathered} \tag{19}$$

In [5, §3] it is proven that (19) has the extremal

$$q(x) = \begin{cases} q_1(x) = -2x + 1\,, & x \in \left[0, \frac{1}{2}\right[\,, \\ q_2(x) = -2x + 2\,, & x \in \left[\frac{1}{2}, 1\right]\,, \end{cases} \tag{20}$$

that is, (20) satisfies the Euler-Lagrange equation (11) for $L(x,q,\dot{q},z) = \frac{1}{3}\left(x+q+z\right)$. We now illustrate the application of our Theorem 13 to this problem. First, we need to determine the variational symmetries. Substituting the Lagrangian L in (16) we obtain that

$$\frac{\tau}{3} + \frac{x+q+z}{3}\dot{\tau} = 0\,. \tag{21}$$

The differential equation (21) admits the solution

$$\tau = ke^{-\int\frac{dx}{x+q+z}}\,, \tag{22}$$

where k is an arbitrary constant. From Theorem 13 we conclude that

$$(x+q_1+z_1)\tau+\frac{1}{3}\int \tau\dot{q}_1 \sum_{t=q_1^{-1}(x)} \frac{1}{|\dot{q}_1(t)|}dx, \quad x\in\left[0,\frac{1}{2}\right[\,, \tag{23}$$

$$(x+q_2+z_2)\tau+\frac{1}{3}\int \tau\dot{q}_2 \sum_{t=q_2^{-1}(x)} \frac{1}{|\dot{q}_2(t)|}dx, \quad x\in\left[\frac{1}{2},1\right]\,, \tag{24}$$

defines a conservation law, where τ is obtained from (22):

$$\tau = ke^{-\int \frac{dx}{3x}} = ke^{\ln x^{-\frac{1}{3}}} = kx^{-\frac{1}{3}},\ x\in[0,1]\,. \tag{25}$$

Since for this problem we know the extremal, we can verify the validity of the obtained conservation law directly from Definition 12: substituting equalities (20) and (25) in (23) and (24), we obtain, as expected, a constant (zero in this case):

$$\begin{aligned}(x+q_1+z_1)\tau+\frac{1}{3}\int \tau\dot{q}_1 &\sum_{t=q_1^{-1}(x)} \frac{1}{|\dot{q}_1(t)|}dx\\ &= 3kx\tau-2\int\tau dx = 3kx^{\frac{2}{3}}-3kx^{\frac{2}{3}}=0\,,\end{aligned}$$

$$\begin{aligned}(x+q_2+z_2)\tau+\frac{1}{3}\int \tau\dot{q}_2 &\sum_{t=q_2^{-1}(x)} \frac{1}{|\dot{q}_2(t)|}dx\\ &= 3kx\tau-2\int\tau dx = 3kx^{\frac{2}{3}}-3kx^{\frac{2}{3}}=0\,.\end{aligned}$$

5 Conclusions

We proved a generalization (i) of the necessary optimality condition of DuBois-Reymond, (ii) of the celebrated Noether's symmetry theorem, for problems of the calculus of variations containing compositions (respectively Theorems 7 and 13). Our main result is illustrated with the example studied in [5].

The compositional variational theory is in its childhood, so that much remains to be done. In particular, it would be interesting to obtain an Hamiltonian formulation and to study more general optimal control problems with compositions.

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